

Framing the structural role of mathematics in physics lectures: A case study on electromagnetism

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DER FORSCHUNG | DER LEHRE | DER BILDUNG

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The interplay of physics and mathematics: A fascinating theme!



Jean Fourier
(1768-1830)

[...] **mathematical analysis is as extensive as nature itself**; it defines all perceptible relations, measures times, spaces, forces, temperatures; [...] It brings together phenomena the most diverse, and **discovers the hidden analogies which unite them** [...] it seems to be a faculty of the human mind destined to **supplement the shortness of life and the imperfection of the senses.**

The interplay of physics and mathematics: Personal motivation

Student trajectory

The Aha-Moment (Why questions) in Math (~ 12)

Similar feeling with Physics! (~ 15)

Civil Engineering – Physics and Mathematics

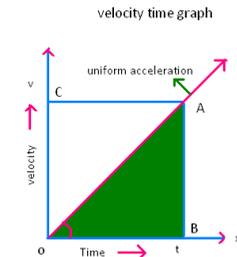
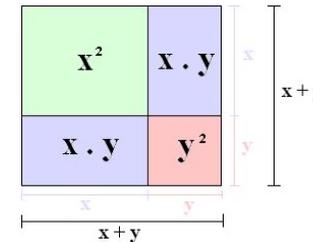
Teaching experience

Mathematics in High School (2000)

Physics in High School (2001)

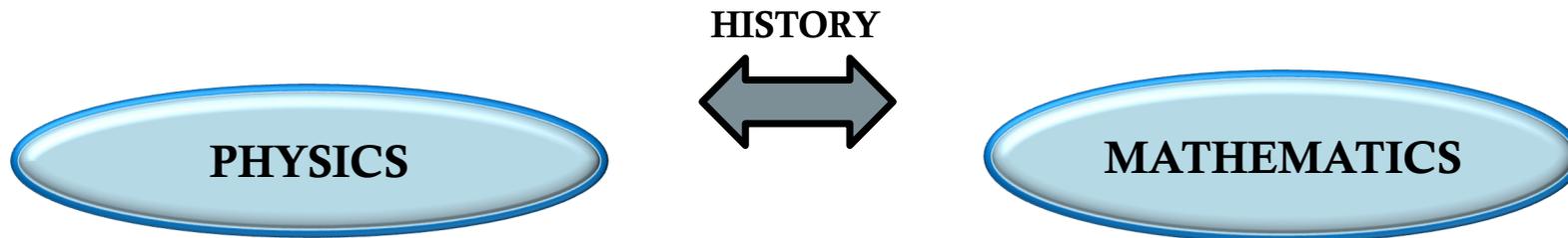
Physics at University (2004)

Mathematics (Calculus and Linear Algebra) at University (2006)



How can mathematics support physical understanding and vice-versa?

Framing the problem: Apparent inconsistency



Mathematics → Physics

- Language
- Reasoning
- Conceptualization
- Formal predictions: Antimatter, E-M Waves

What did mathematics do to physics? (Gingras, 2001)

Physics → Mathematics

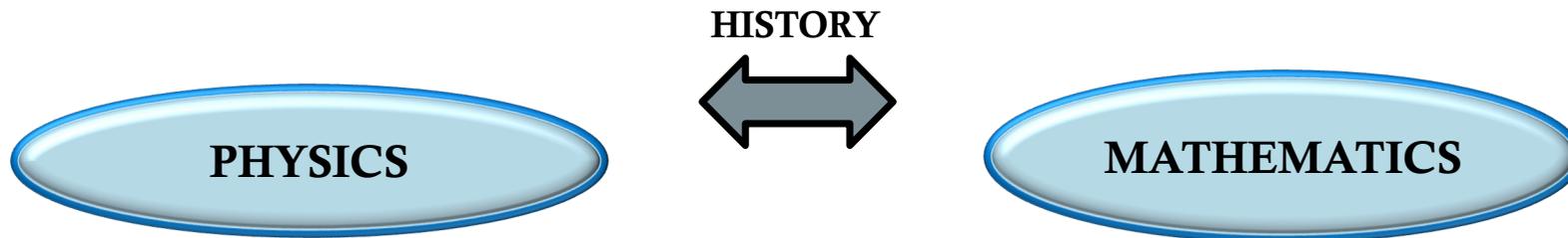
The desire to understand nature has had the most constant and happiest influence on the development of mathematics. In the first place the physicists set us problems whose solution they expect of us. But in proposing these problems, they have largely paid us in advance for the service we shall render them, if we manage to solve them.



Henri Poincaré

(1854-1912)

Framing the problem: Apparent inconsistency



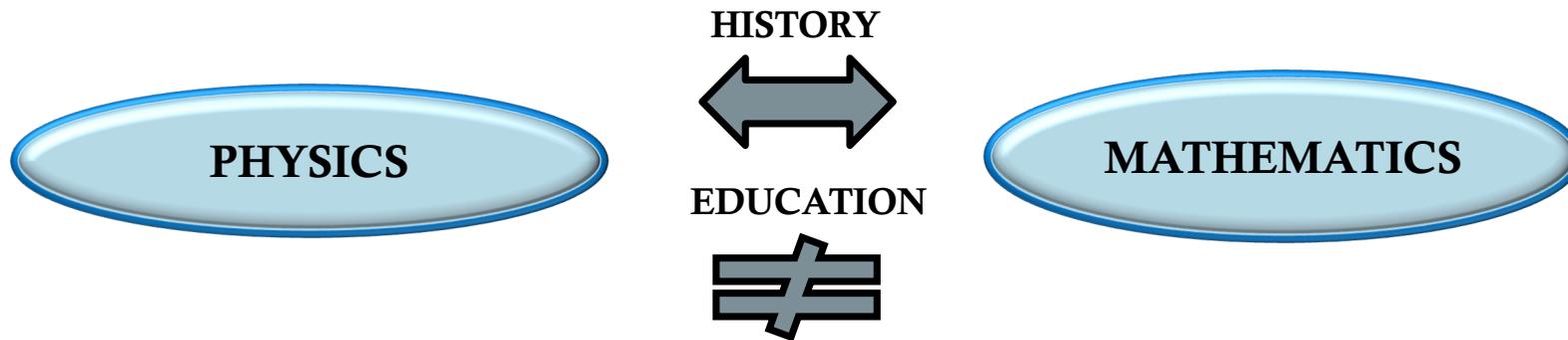
Mathematics \rightarrow Physics

- Language
- Reasoning
- Conceptualization
- Formal predictions: Antimatter, E-M Waves

Physics \rightarrow Mathematics

- Understanding nature
- Source of problems
- Motivation: Calculus, Vector Algebra, Fourier Analysis

Framing the problem: Apparent inconsistency



- Physics' classes: Math is a calculation tool
- Math's classes: Physics is an application of math

The role(s) of mathematics in physics:

Technical and Structural dimensions

<i>Technical</i> (instrumental, procedural)	<i>Structural</i> (relational, organizational)
Blindly use equations to solve quantitative problems (plug-and-chug)	Derive an equation from physical principles using logical reasoning
Focus on algorithmic manipulations	Focus on physical interpretations
Use arguments of authority ; Rote memorization of equations and rules	Justify the use of specific mathematical structures to model physical phenomena
Fragmented knowledge : memorize different equations for each specific case	Structured knowledge : connect apparently different physical assumptions through logic
Identify superficial similarities between equations (e.g. $d = 1/2 gt^2$ and $K = 1/2 mv^2$)	Recognize profound analogies and common mathematical structures behind different physical phenomena (e.g. Central force field)
Mathematics seen as calculation tool	Mathematics seen as reasoning instrument
Mathematics as “just another” language to represent and communicate	Mathematics is essential to define physical concepts and structure physical thought

Case-study: Structural dimension in physics lectures

PhD at USP and TU Dresden – PR-STPER (in press)

- Excellent reputation und evaluation
- Emphasize the structural role of math
- Didactic strategies



Lectures filmed

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- Relativity and Electromagnetism
- Total of 70h of data
- Focus on the use of math
- 20 Episodes (concepts and problems)
- Analysis with *Videograph* (20s)
- Semi-structured interview

Categories of analysis

Mathematizing

Interpreting

Technical

Visual

Analogy

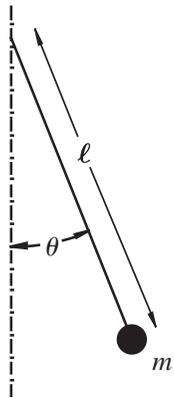
Deduction

Philosophy

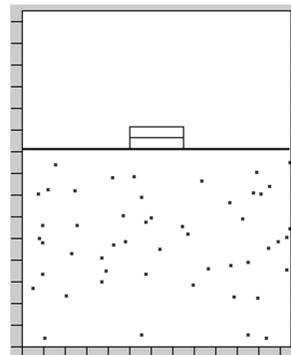
Metacognition

Mathematizing (from Physics to Math)

M1 - Assumptions: approximate, idealize, relevant variables



- No damping (friction/air)
- Infinitely light rigid rod
- Point mass
- Small angles

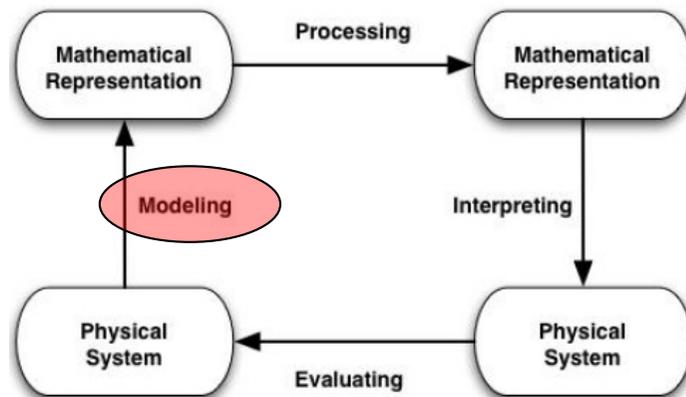


- Point masses
- Random movement
- Elastic collisions
- No interaction

You can **approximate** one road as a one-dimensional thing. Its length is much bigger than its width. One strand of hair, **you can think of it as a one-dimensional thing**. Does it have only one dimension? No! It has 3.

Mathematizing (from Physics to Math)

M2 - Use mathematical structures (“Prefabricated math”)



Physical aspect → Math structure

Periodic → Trig. functions

Direction → Vectors

Projection → Scalar product

Order matters → Matrices

In order to maintain the symmetries of space and time this rule **has to be a linear transformation.**

Interpreting (from Math to Physics)

Assigning physical meaning to mathematical expressions
(what does it say/mean?)

Analyzing special and limit cases

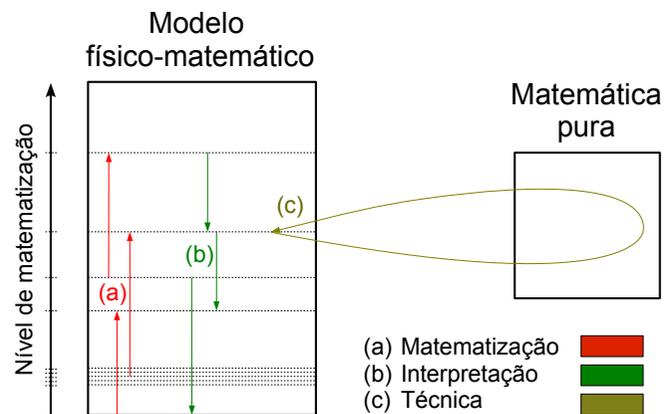
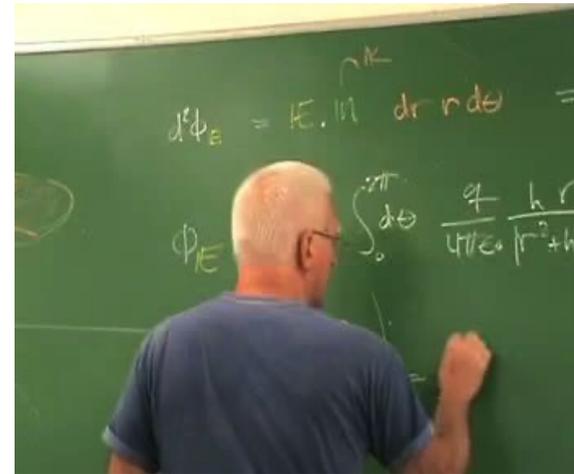
What do you (physically) expect from the result? Test/play with the result. What if x , y or $z = 0$? Is it reasonable? **If you don't expect anything from an equation you're lost.**

Technical (manipulations and authority)

Algorithmic manipulations

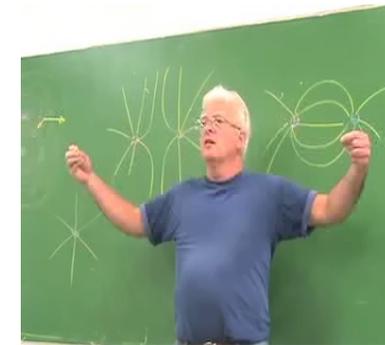
Arguments of authority

Explanation of the technique

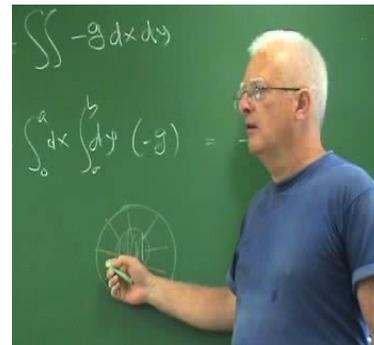


Visual representations

V1 - Gestural



V2 - Pictorial



Analogical reasoning

A1 - Everyday life examples/physical material analogies

Where can we find gradient in nature? [...] Weather, pressure, heat...
Can dogs calculate the gradient? How can a dog find a steak inside a dark room?

A2 - Recognizing formal analogies

Today we learned some strategies to deal with distributions of things.
This is very general, it can be with charge, mass, population, anything that needs to be distributed.

Deductive reasoning

PROPOSITION 19.

In any triangle the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than the angle BCA ;

I say that the side AC is also greater than the side AB .

For, if not, AC is either equal to AB or less.

Now AC is not equal to AB ;

for then the angle ABC would also have been equal to the angle ACB ;

[I. 5]

but it is not;

therefore AC is not equal to AB .

Neither is AC less than AB ,

for then the angle ABC would also have been less than the angle ACB ;

[I. 18]

but it is not;

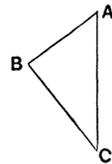
therefore AC is not less than AB .

And it was proved that it is not equal either.

Therefore AC is greater than AB .

Therefore etc.

Q. E. D.



Euclid's Elements (circa 300 BC)

Let CB be any horizontal line and AB an inclined plane; after descent along AB let the motion continue through the assigned horizontal distance BD . Then,

I say, the time of descent along AB bears to the time spent in traversing BD the same ratio which twice AB bears to BD .

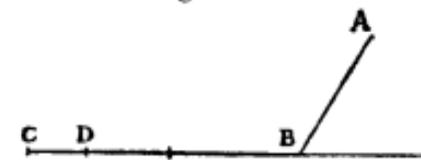


Fig. 86

For, lay off BC equal to twice AB then it follows, from a previous proposition, that the time of descent along AB is equal to the time required to traverse BC ;

but the time along BC is to the time along DB as the length CB is to the length BD . Hence the time of descent along AB

[247]

is to the time along BD as twice the distance AB is to the distance BD .

Q. E. D.

Galilei's Two New Sciences (1638)

S: I'm not convinced that these components should be equal to those. Isn't there a more rigorous way of proving that?

P: Can you demonstrate this equality from a mathematical point of view? The answer is no!

Philosophy

In physics we use mathematics in a very intuitive way. **The mathematician thinks differently**, but as physicists our thought is very related to the material world.

This (reference of frame) is a reasoning instrument, **it is not in nature, but in your mind. It is an invention.**

Metacognition

Think and **ask yourself if you really understood the problem**, not more or less

Wait a minute. Now I'm using a different strategy to cut the body. **Do you think it is better to do it like this?** Does the answer depend on the strategy

Videograph interface: Coding variables

Videograph - Análise aula 5 - Descrição matemática da densidade de cargas - TESE.VDG

File View Window Transcript Coding Options ?

Video 1

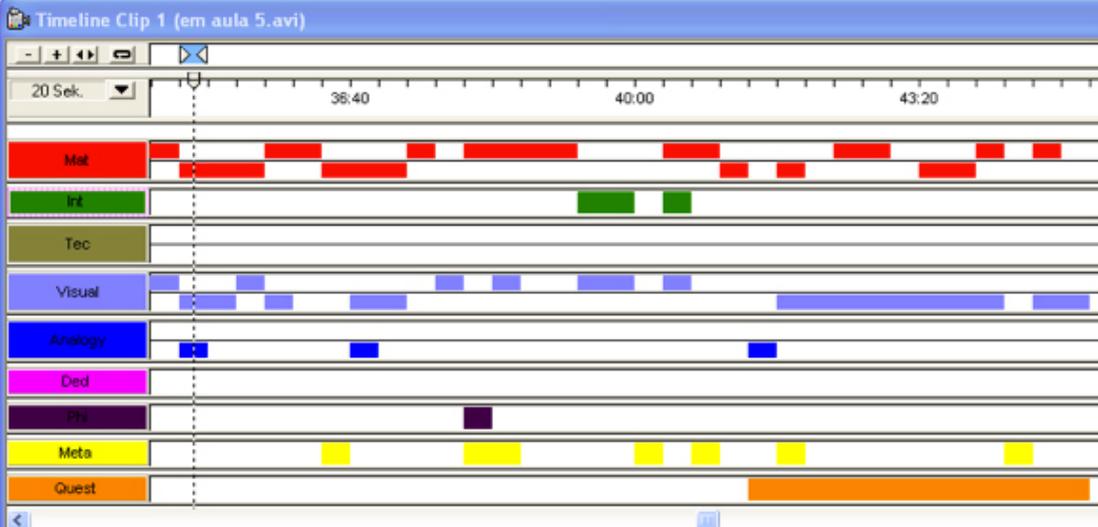


Transcript mediaclip 1

Mat	2	Representing physical magnitudes with mathematical structures/prefabricated math
Int	1	Modeling (idealize/approximate/select relevant variables)
Tec	2	Assigning physical meaning to mathematical expressions
Visual	1	Conceptual understanding
Analogy	1	Manipulation and Authority
Ded	2	Pictorial/graphical representations
Phi	1	Gestures
Meta	2	Highlighting formal similarities and differences
Quest	1	Everyday life examples/physical material analogies/metaphors
	1	Mention deductive aspects (general principles, prove, show...)
	1	Explicit philosophical remarks (relations between physics and mathematics)
	1	Knowledge about own thought and self regulation
	1	Question and Answer

Timeline Clip 1 (em aula 5.avi)

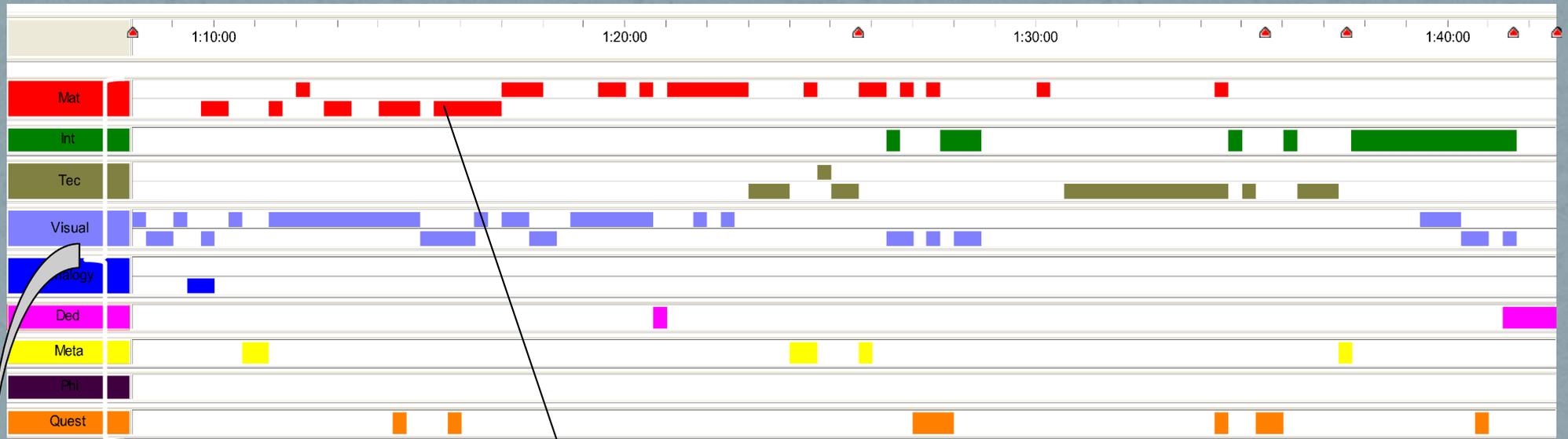
20 Sek. 36:40 40:00 43:20



0:34:40 - 0:35:00

O que eu estou falando é o seguinte: na hora que você for contar elétrons, o elétron é discreto. Mas imagina que essa fila aqui tenha 80 km. De gente distribuída numa fila (gestos). Numa escala dessa eu vou abandonar essa ideia de falar de um por um.

Episode 16: Electric flux through a disk (25 min)

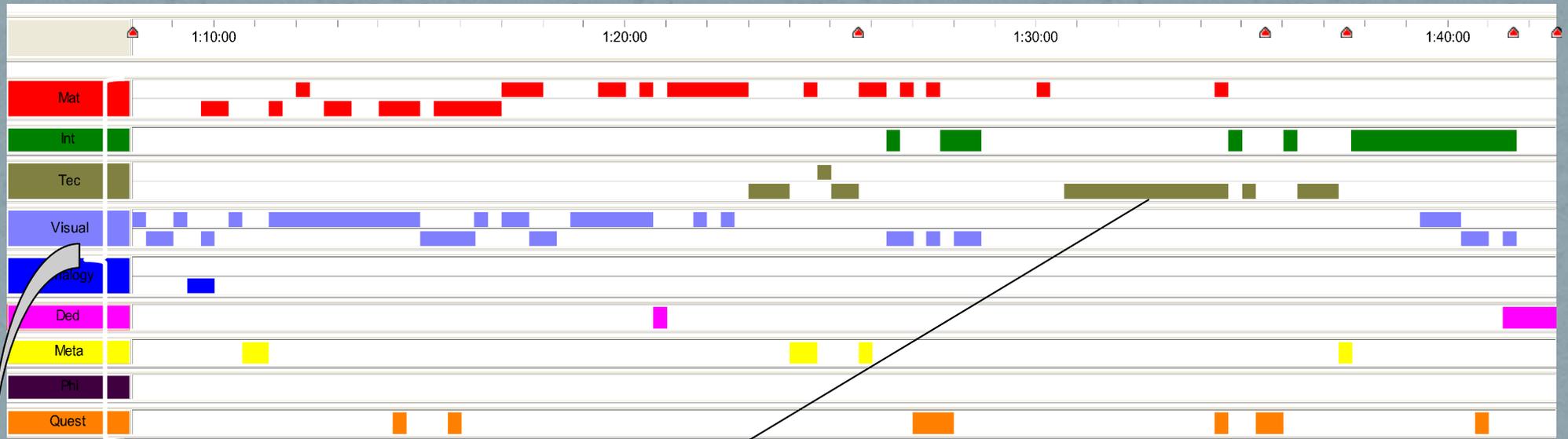


- Mat
- Int
- Tec
- Vis
- Ana
- Ded
- Epist
- Meta
- Quest

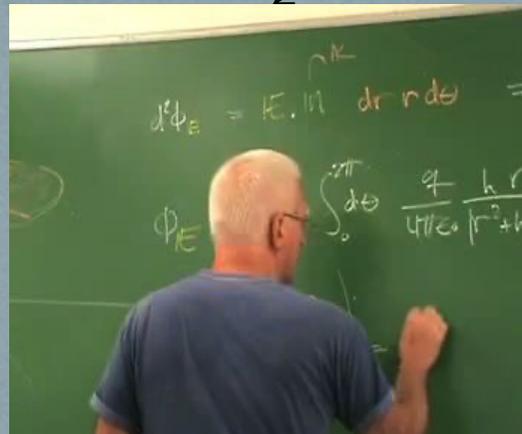


The situation now is more complex because for each dS you have a different flux [...] Consider a surface element here (z) and another here (aside), which one has a greater flux? Why?

Episode 16: Electric flux through a disk (25 min)



- Mat*
- Int*
- Tec*
- Vis*
- Ana*
- Ded*
- Epist*
- Meta*
- Quest*

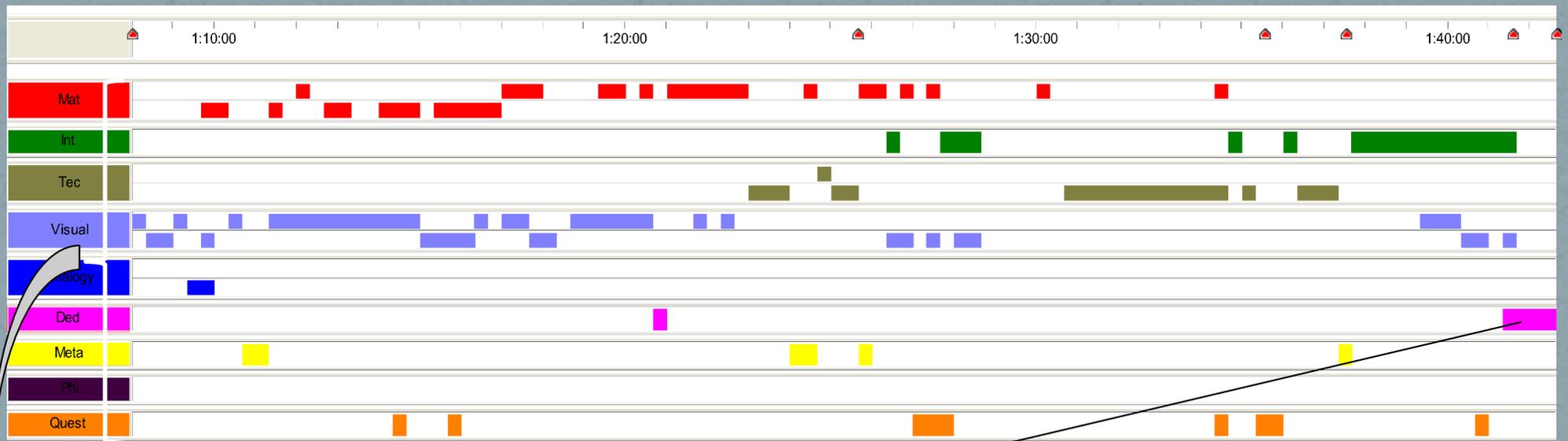


$$\vec{E}_P = \frac{q}{4\pi\epsilon_0} \frac{(\vec{r}_P - \vec{r}_q)}{|\vec{r}_P - \vec{r}_q|^3} \text{ com } \vec{r}_q = 0 \text{ e } \vec{r}_P = r\cos\theta\hat{i} + r\sin\theta\hat{j} + h\hat{k}$$

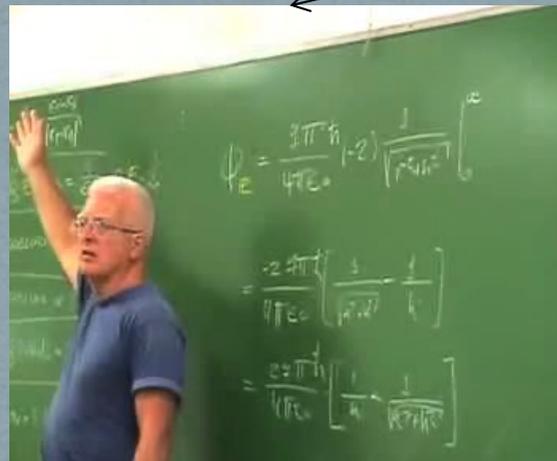
$$d^2\Phi_{\vec{E}} = \vec{E} \cdot \hat{n} dS = \left[\frac{q}{4\pi\epsilon_0} \frac{r\cos\theta\hat{i} + r\sin\theta\hat{j} + h\hat{k}}{(r^2 + h^2)^{3/2}} \right] \cdot [rdrd\theta\hat{k}] = \frac{q}{4\pi\epsilon_0} \frac{rhdrd\theta}{(r^2 + h^2)^{3/2}}$$

$$\Phi_{\vec{E}} = \int_0^a dr \int_0^{2\pi} d\theta \frac{q}{4\pi\epsilon_0} \frac{rhdrd\theta}{(r^2 + h^2)^{3/2}} = \frac{q}{2\epsilon_0} \left[1 - \frac{1}{\sqrt{1 + a^2/h^2}} \right]$$

Episode 16: Electric flux through a disk (25 min)



- Mat
- Int
- Tec
- Vis
- Ana
- Ded
- Epist
- Meta
- Quest



Gauss's law inverts the order of these things. It assumes the flux constancy as a greater truth than that one (Coulomb). [...] What do I learn when I say that this is more important? Now I know WHY the field is proportional to $1/r^2$.

Conclusions & Perspectives

- Analytical tool: Categories to classify different “roles” of mathematics in physics instruction
- Case study: From concrete to abstract; Long mathematization; Analogies; Metacognition (it is not easy); Philosophy
- Hypothesis: Structural dimension demands colorfulness (?)
- Compare with other Professors; Teachers; Textbooks (?)
- Assess impact on learning (tools?)