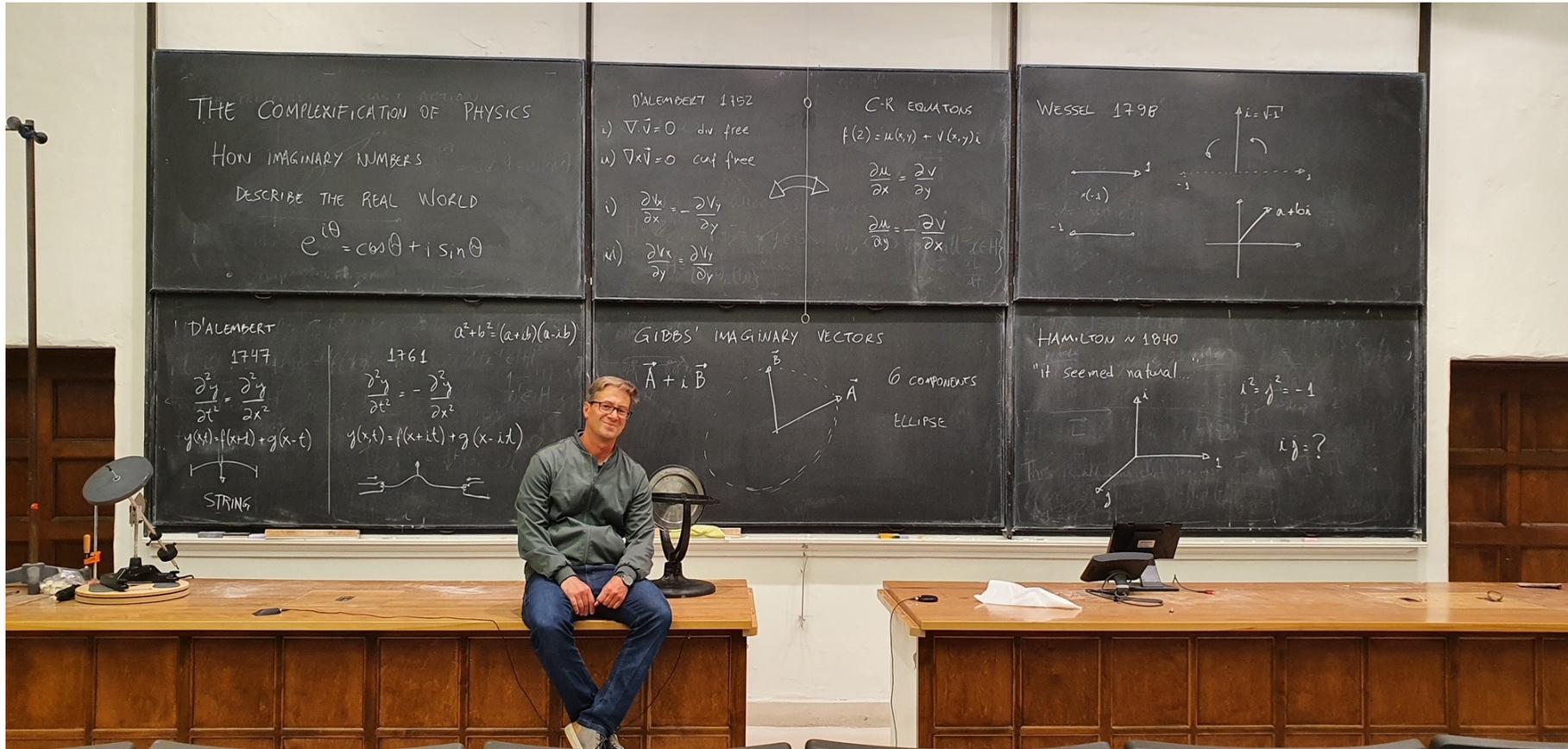# The **complexification** of physics:

Historical episodes and educational implications

Ricardo Karam

UNIVERSITY OF COPENHAGEN

An old dream project...


Feynman Lecture Hall – CALTECH 2022

PRINCETON UNIVERSITY PRESS

Motivation

Complex numbers have been known to physics for two hundred years, and in the nineteenth century their occurrence in physics has gradually widened so that a veritable ***complexification of physics*** has come about. However, there seems to be **no recognized topic** in philosophy whose subject matter is to evaluate the significance of complex numbers in mathematics, physics, and science in general (Bochner, 1963).
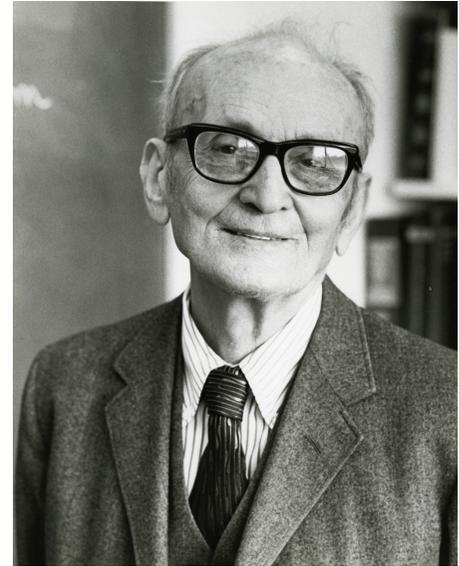
Salomon Bochner
(1899-1982)

The Role of Mathematics in the Rise of Science

By Salomon Bochner
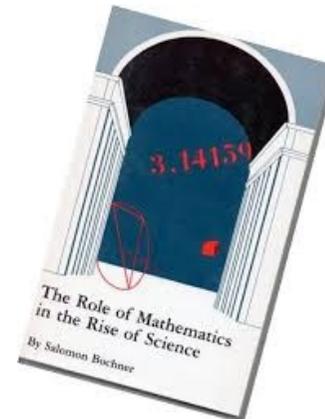
Outline

- Brief historical sketch

- Episode 1: Circular polarization from complex numbers

- Episode 2: Quaternions and Electromagnetism

- Episode 3: CR equations and Hydrodynamics

- Episode 4: Schrödinger's real component of ψ

16th century: Origins

Cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

Del Ferro (1515)
Tartaglia (1535)
Cardano (1545)
Bombelli (1572)

ITALIA 750

$$x^3 = px + q$$

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

$$\frac{q^2}{4} - \frac{p^3}{27} < 0$$

casus irreducibilis

$$x^3 = 15x + 4$$

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

$$[+\sqrt{-n} \times +\sqrt{-n} = -n]$$
$$[+\sqrt{-n} \times -\sqrt{-n} = +n]$$
$$[-\sqrt{-n} \times +\sqrt{-n} = +n]$$
$$[-\sqrt{-n} \times -\sqrt{-n} = -n]$$

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$$

$$\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}$$

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

- √-1 useful "trick" to find (real) roots (cancel out)
- a + bi + c + di = (a + c) + (b + d)i
- (a + bi)(c + di) = (ac – bd) + (ad + bc)i

UNIVERSITY OF COPENHAGEN

19th century: Geometrical interpretation

Analytical representation of direction (Wessel, 1798)

- *Change in direction should be represented by algebraic symbols*

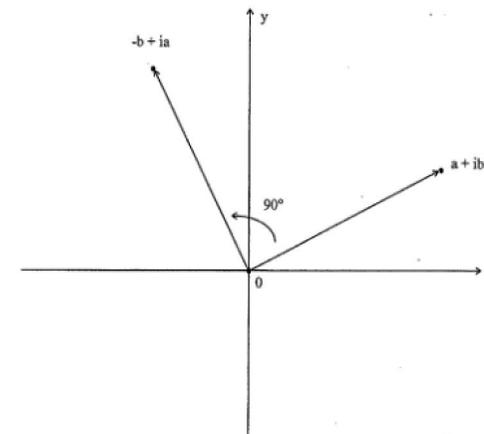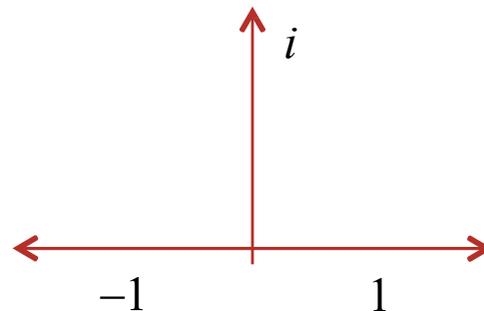- *Problem: direction cannot be changed by algebraic operations except to the opposite*

.(-1)
means 180º

-1 1

How to express 90º?

Hint: $i^2 = -1$

-1 1

i
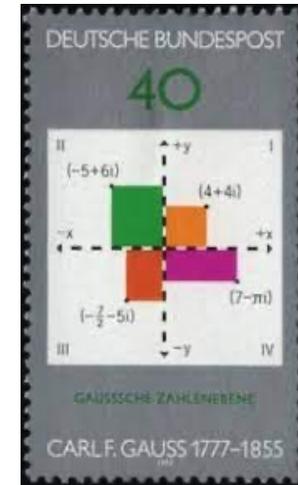
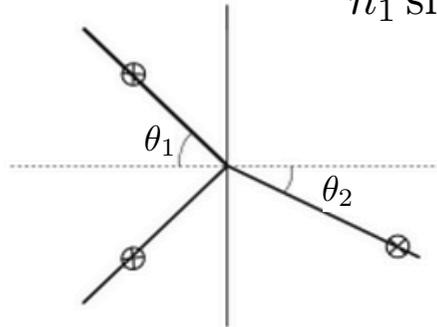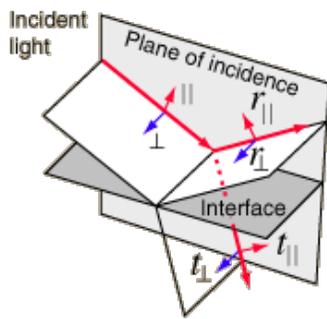*Multiply by i means 90º CCW rotation!*

$$v = a + bi$$

Fresnel equations become complex

MÉMOIRE

SUR LA LOI DES MODIFICATIONS

QUE LA RÉFLEXION IMPRIME A LA LUMIÈRE POLARISÉE [*],

LU À L'ACADÉMIE DES SCIENCES, LE 7 JANVIER 1823.

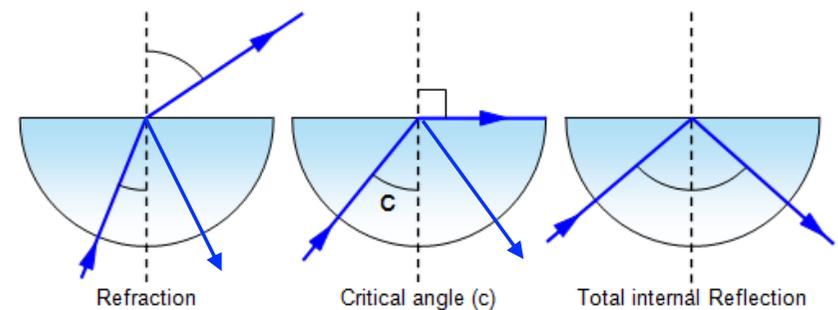$$r_\perp = \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_1 + \theta_2)}$$

$$\boxed{r_\perp = \frac{n\cos(\theta_1) - \sqrt{1 - n^2 \sin^2(\theta_1)}}{n\cos(\theta_1) + \sqrt{1 - n^2 \sin^2(\theta_1)}}}$$

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

$$\frac{n_1}{n_2} = n$$

When $n^2 \sin^2(\theta_1)$

$$\begin{cases} < 1 & |r_\perp| < 1 \\ = 1 & r_\perp = 1 \\ > 1 & r_\perp \text{ is not real!} \end{cases}$$
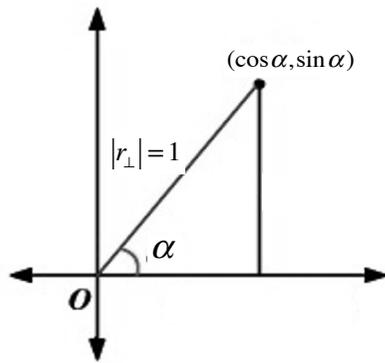
Due to the *general law of continuity*, if there is an accurate expression for the laws of reflection just before the limit, it should remain valid afterwards; *the challenge is to interpret/guess what analysis says about these imaginary expressions.*

Refraction   Critical angle (c)   Total internal Reflection

$$r_\perp = \frac{n\cos(\theta_1) - \sqrt{1 - n^2\sin^2(\theta_1)}}{n\cos(\theta_1) + \sqrt{1 - n^2\sin^2(\theta_1)}}$$

Problem: How to measure this phase shift?

Solution: Look for the *difference* in the phase shifts of the $\perp$ and $\parallel$ components

$$\sqrt{1 - n^2\sin^2(\theta_1)} = \sqrt{n^2\sin^2(\theta_1) - 1}\cdot\sqrt{-1}$$

$$r_\perp = |r_\perp|[\cos\alpha + i\sin\alpha]$$

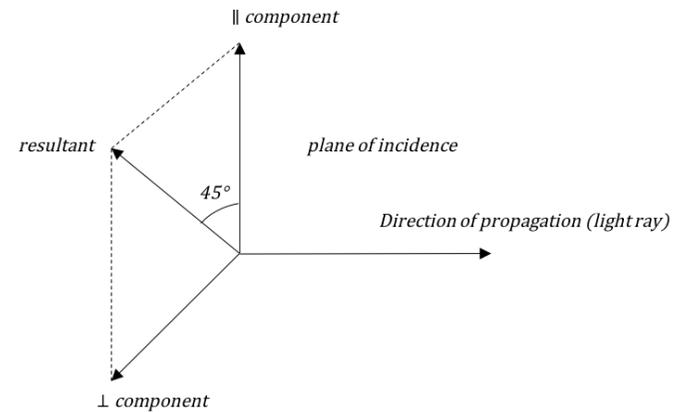$(\cos\alpha, \sin\alpha)$

$|r_\perp| = 1$

α

O

It means without a doubt that the periods of vibrations[15] of the reflected waves, which in the basis of the calculations were supposed coincident at the surface with the ones from the incident waves, no longer coincide [...] these periods are retarded or advanced by a certain quantity.

Interpretation:
Phase shift!

‖ component

resultant   plane of incidence
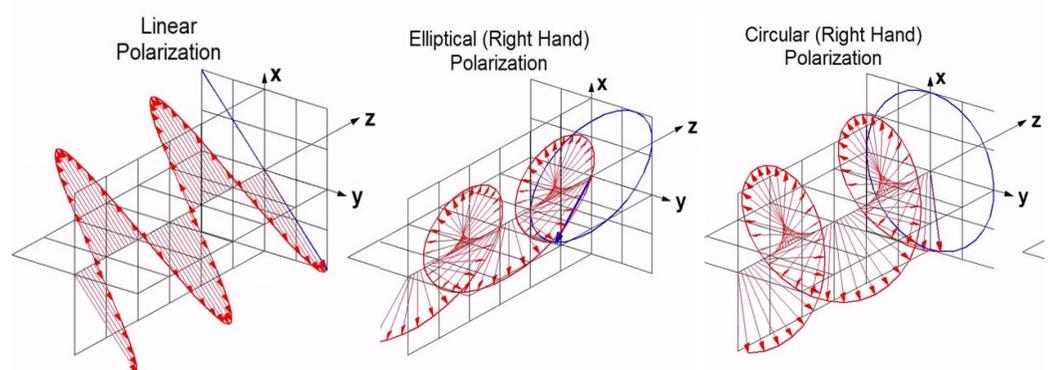
45°

Direction of propagation (light ray)

⊥ component

⊥ and ‖ in phase   ⊥ and ‖ out of phase   ⊥ and ‖ phase difference = 90°

Linear
Polarization

Elliptical (Right Hand)
Polarization

Circular (Right Hand)
Polarization

The waves are superposed. The resultant wave lies in a plane.

https://www.youtube.com/watch?v=Fu-aYnRkUgg

UNIVERSITY OF COPENHAGEN

‖ component

plane of incidence

resultant

45°

Direction of propagation (light ray)

⊥ component

Circular polarization

⊥ and ‖ phase difference = 90°
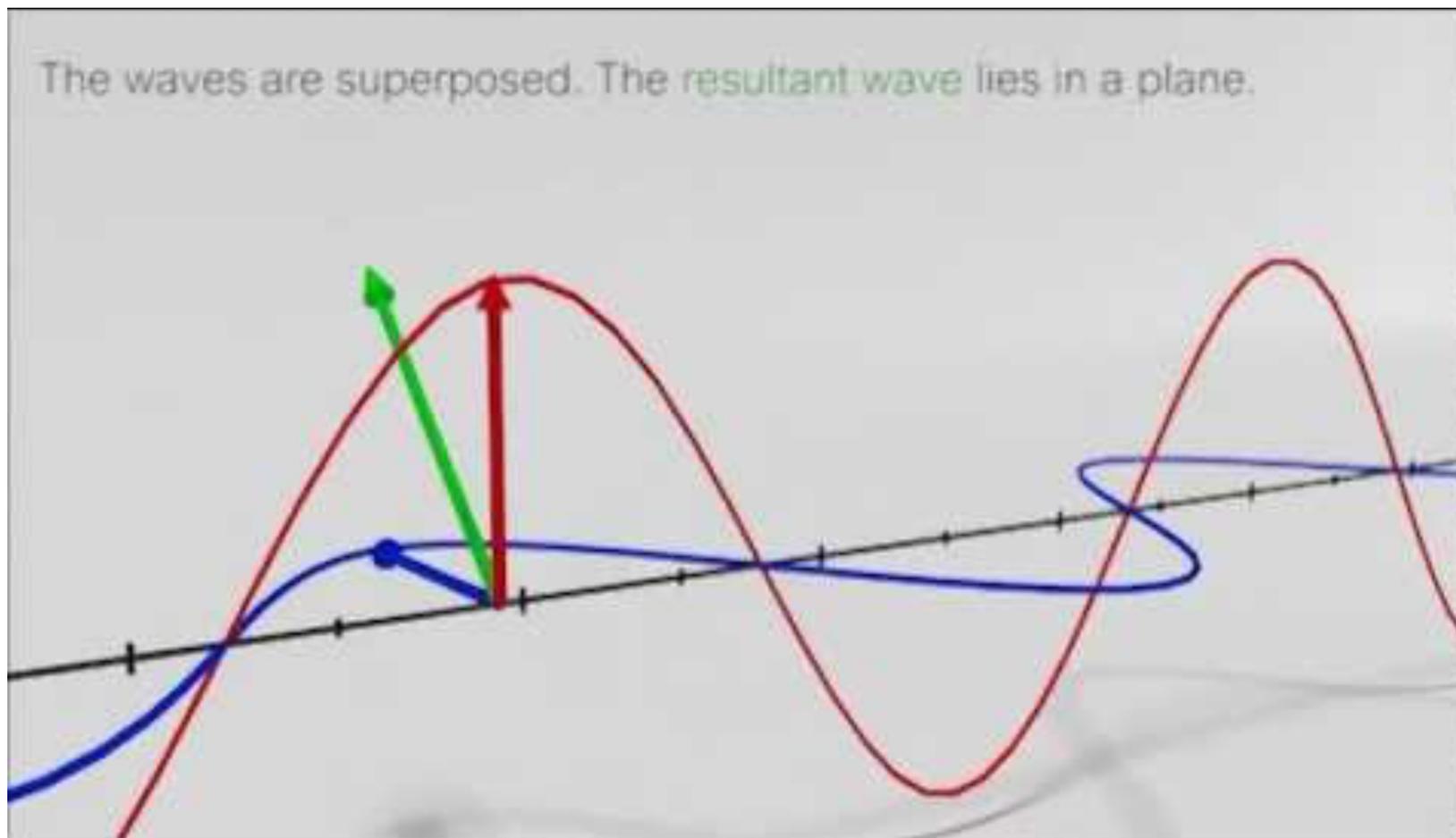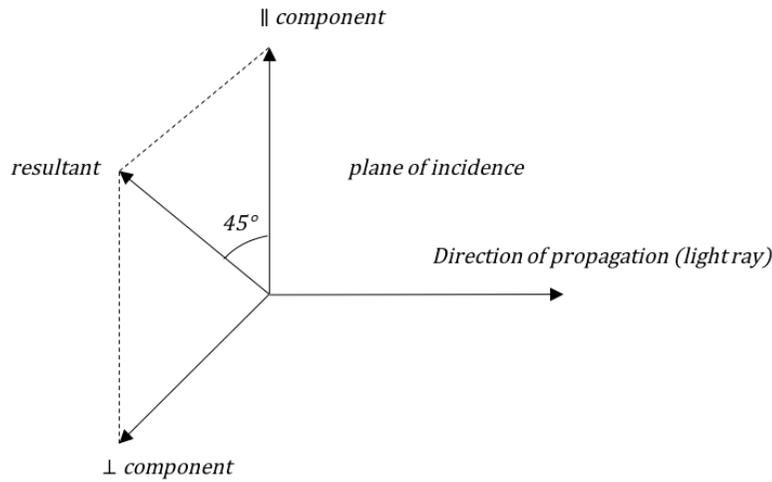
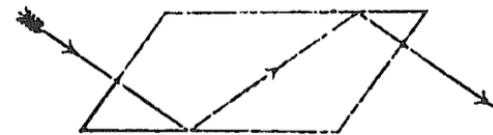$$\cos(\alpha - \beta) = \frac{2n^2 \sin^4\theta_i - (n^2 + 1)\sin^2\theta_i + 1}{(n^2 + 1)\sin^2\theta_i - 1}$$

Fresnel Rhomb

$\theta_i = 54°37'$

3 reflections ($\alpha - \beta = 30°$)

$\theta_i = 43°10'$

$\theta_i = 69°12'$

4 reflections ($\alpha - \beta = 22.5°$)

$\theta_i = 74°42'$

β

α

α − β

θ_i
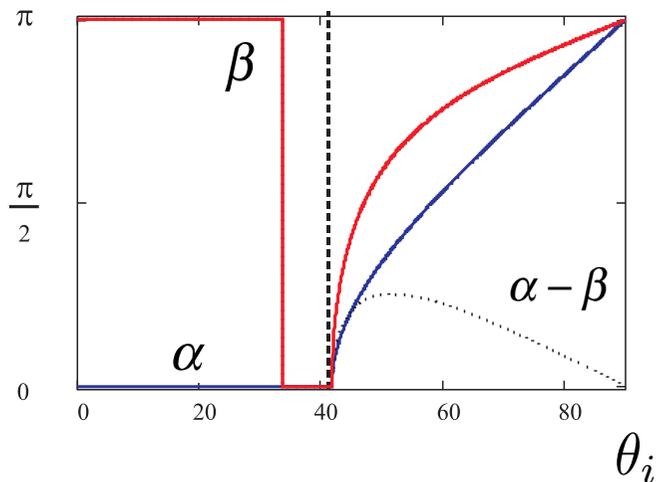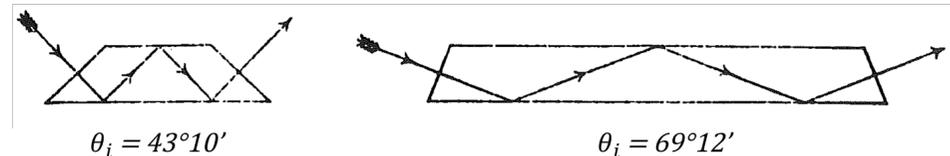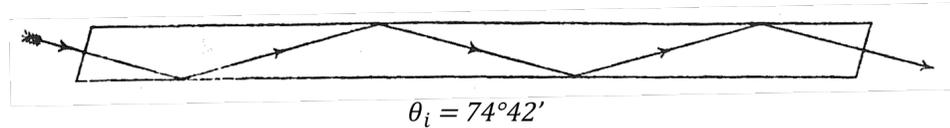
Fresnel's original interpretation of complex numbers in 19th century optics

EP

Ricardo Karam[a]

Some lessons from Episode 1

Fresnel (1823) - **Physical interpretation** (phase shift) of complex numbers

- *One can say that this was the first time in which "nature" was abstracted from "pure" mathematics, that is from a mathematics which had not been previously abstracted from nature itself (Bochner, 1963)*

- Circularly polarized light is "man-made" and due to complex numbers

Treatise on Electricity and Magnetism - Maxwell (1873)

$$4\pi\mu u = \frac{dJ}{dx} + \nabla^2 F.$$
$$4\pi\mu v = \frac{dJ}{dy} + \nabla^2 G,$$
$$4\pi\mu w = \frac{dJ}{dz} + \nabla^2 H.$$

A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities, which are completely represented by one numerical quantity, and Vectors, which require three numerical quantities to define them.

The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple co-ordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$
$$b = \frac{dF}{dz} - \frac{dH}{dx},$$
$$c = \frac{dG}{dx} - \frac{dF}{dy}.$$

Quaternion Expressions for the Electromagnetic Equations.

618.] In this treatise we have endeavoured to avoid any process demanding from the reader a knowledge of the Calculus of Quaternions. At the same time we have not scrupled to introduce the idea of a vector when it was necessary to do so. When we have had occasion to denote a vector by a symbol, we have used a German letter, the number of different vectors being so great that Hamilton's favourite symbols would have been exhausted at once. Whenever therefore, a German letter is used it denotes a Hamiltonian vector, and indicates not only its magnitude but its direction.

619.] The equations (A) of magnetic induction, of which the first is,

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

may now be written

$$\mathfrak{B} = V \nabla \mathfrak{A},$$

where ∇ is the operator

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

and V indicates that the vector part of the result of this operation is to be taken.

Since $\mathfrak{A}$ is subject to the condition $S\nabla\mathfrak{A} = 0$, $\nabla\mathfrak{A}$ is a pure vector, and the symbol V is unnecessary.

What are quaternions? How are they related to vectors?

Analytical representation of direction – Wessel (1798)

Change in direction should be represented by algebraic symbols

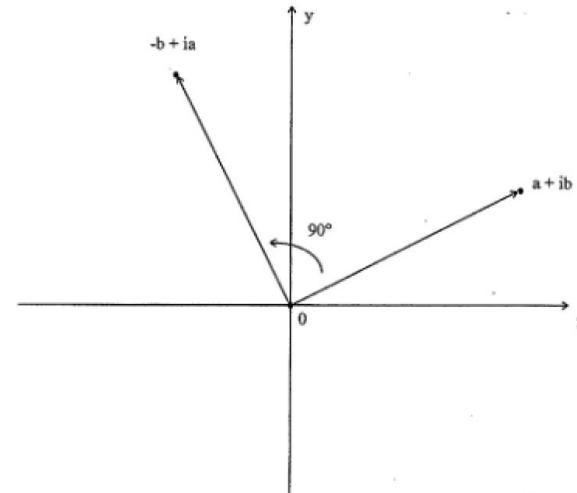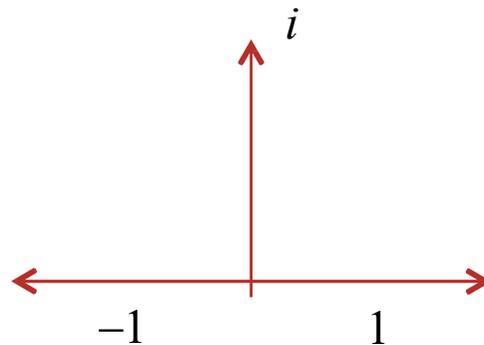*Problem: direction cannot be changed by algebraic operations except to the opposite*

.(-1)
means 180º

$-1 \qquad 1$

How to express 90º?

Hint: i² = -1

i

$-1 \qquad 1$

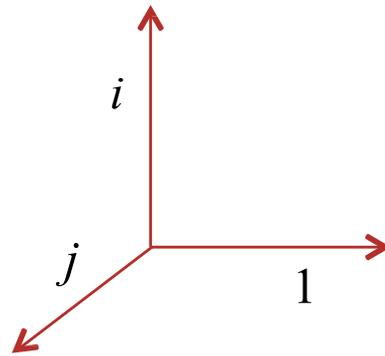-b + ia

y

90º

a + ib

0

x

i means 90º rotation!

$$v = a + bi$$

Analytical representation of direction in 3D – Hamilton (1843)

Hamilton's path to quaternions

A plausible 2D-3D analogy: "it seemed natural"

Triplet

Since i is in a certain well-known sense, a line perpendicular to the line 1, it seemed natural that there should be some another imaginary to express a line perpendicular to the former.

$$j^2 = i^2 = -1$$

Neves (2008)

$$x + iy + jz$$

… I tried the effect of assuming also $j^2 = -1$, which I interpreted as a rotation through two right angles in the xz, as $i^2 = -1$ had corresponded to such a rotation in the plane xy
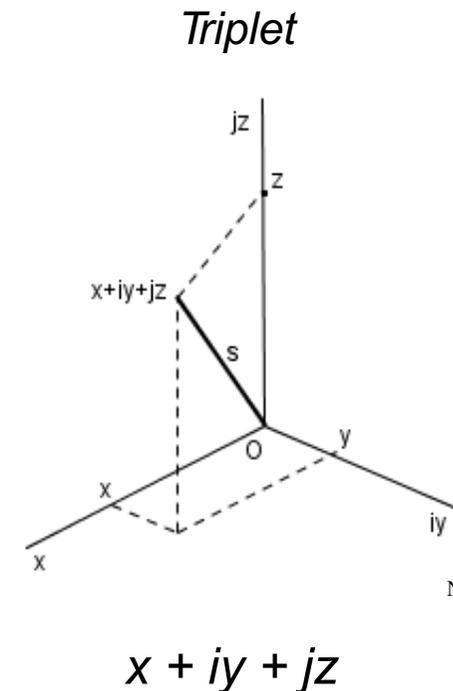
Analytical representation of direction in 3D – Hamilton (1843)

Addition of triplets

$$(a + ib + jc) + (x + iy + jz) = (a + x) + i(b + y) + j(c + z) \qquad \checkmark$$

Multiplication of triplets

$$(a + ib + jc)(x + iy + jz) = ax + aiy + ajz + ibx + ibiy + ibjz + jcx + jciy + jcjz$$
$$= ax + i^2by + j^2cz + iay + ibx + jaz + jcx + ijbz + ijcy$$
$$= (ax - by - cz) + i(ay + bx) + j(az + cx) + ij(bz + cy) \qquad ?$$

But what are we to do with ij? Shall it be of the form $\alpha + i\beta + j\gamma$?

1st attempt: $ij = \pm 1$

2nd attempt: $ij = 0$

3rd attempt: $ij = -ji = k$

*Details in Hankins (1980, p. 295-300)*

Analytical representation of direction in 3D – Hamilton (1843)

Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

& cut it on a stone of this bridge

Neves (2008)

Thus in a very dramatic manner Hamilton discovered and announced the discovery of quaternions. These are hypercomplex numbers of the form $w + ix + jy + kz$, where w, x, y, and z are real numbers, and i, j, and k are unit vectors, directed along the x, y, and z axes respectively. The i, j, and k units obey the following laws:

$$ij = k \qquad jk = i \qquad ki = j$$
$$ji = -k \qquad kj = -i \qquad ik = -j$$
$$ii = jj = kk = -1$$
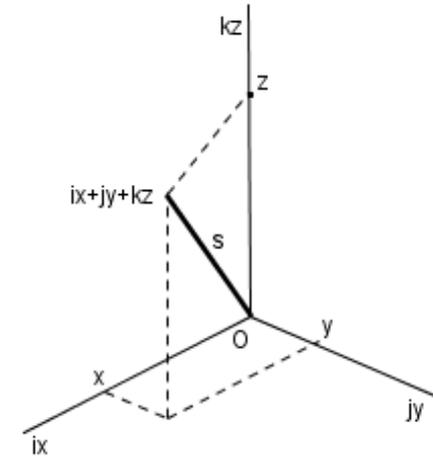
$$(ix + jy + kz)(ix' + jy' + kz') = i^2 xx' + ijxy' + ikxz' + jiyx' + j^2 yy' + jkyz' + kizx' + kjzy' + k^2 zz'$$
$$= \dots = (-xx' - yy' - zz') + i\underbrace{(yz' - zy')}_{x''} + j\underbrace{(zx' - xz')}_{y''} + k\underbrace{(xy' - yx')}_{z''}$$
$$= (-xx' - yy' - zz') + ix'' + jy'' + kz''.$$

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

Vector algebra – Heaviside (& Gibbs)

Maxwell exhibited his main results in quaternionic form. I went to Prof. Tait's treatise to get information, and to learn how to work them. […] But on proceeding to apply quaternions to the development of electric theory, I found it very inconvenient. Quaternions were in their vectorial aspects antiphysical and unnatural […]. So I dropped out the quaternion altogether, and kept to pure scalars and vectors, using a very simple vectorial algebra in my papers from 1883 onwards.

Heaviside (1893)

Quotation from Crowe (1967)

Some lessons from Episode 2

- Why do we call the unit vectors i, j and k ? They were originally imaginary units!

- Scalars and vectors were parts of a whole (quaternions). Quaternions were created to represent direction analytically in 3D.

- Vectors originate in an attempt to improve the use of quaternions in EM, which resulted in "getting rid" of them...

Curious similarity

2D fluid flow

Complex derivative

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$
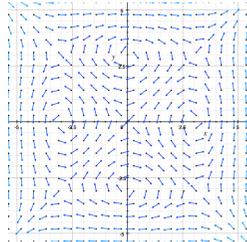
$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Incompressible

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\boxed{\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

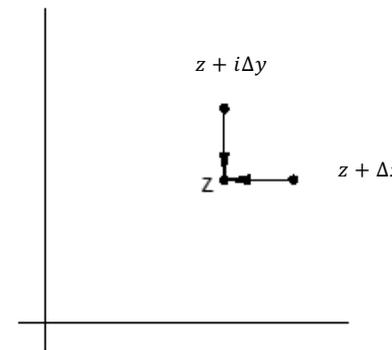$z + i\Delta y$

z $z + \Delta x$

Irrotational

$$\nabla \times \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

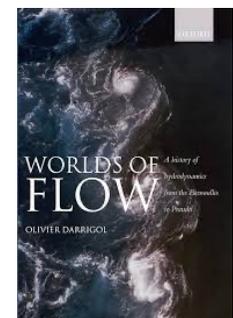$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}}$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

CR equations

Fig. 13
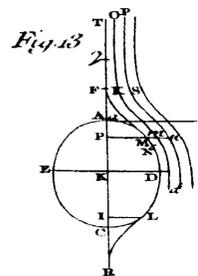
E S S A I
D'UNE
NOUVELLE THEORIE
DE LA
RÉSISTANCE DES FLUIDES.
Par M. D'ALEMBERT, de l'Académie Royale des Sciences
de Paris, de celle de Prusse, & de la Société Royale de Londres.

"d'Alembert discovered the Cauchy–Riemann condition for u and −v to be the real and imaginary components of an analytic function in the complex plane (Darrigol, 2005)

WORLDS OF
FLOW
OLIVIER DARRIGOL

Should the CR equations be named after d'Alembert?

D'Alembert's "ingenious method" (1752)

Incompressible

exact differential

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \Longleftrightarrow \quad vdx - udy \text{ is an exact differential} \qquad f(x,y) \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Irrotational

so is $\quad ivdx - iudy = ivdx + u\frac{dy}{i}$ (2)

equality of mixed partials

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \Longleftrightarrow \quad udx + vdy \text{ is an exact differential} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

so is $\quad udx + iv\frac{dy}{i}$ (1)

$(1) + (2) \quad (u+iv)\left(dx + \frac{dy}{i}\right) \quad$ is **also** an exact differential
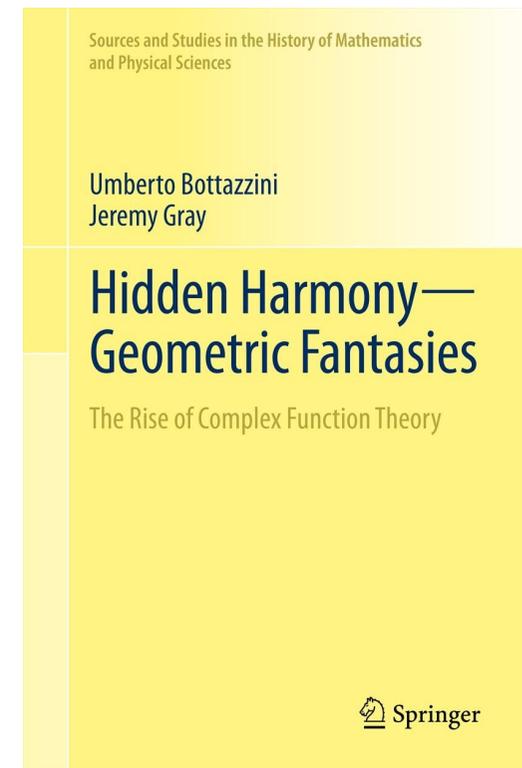
Thus, $u + iv$ is a **function** of $x + \frac{y}{i}$

$(1) - (2) \quad (u-iv)\left(dx - \frac{dy}{i}\right)$

$$u + iv = \phi\left(x + \frac{y}{i}\right) + i\psi\left(x + \frac{y}{i}\right) \qquad u = \frac{1}{2}\left[\phi\left(x + \frac{y}{i}\right) + \phi\left(x - \frac{y}{i}\right)\right] + \frac{i}{2}\left[\psi\left(x + \frac{y}{i}\right) - \psi\left(x - \frac{y}{i}\right)\right]$$

$$u - iv = \phi\left(x - \frac{y}{i}\right) - i\psi\left(x - \frac{y}{i}\right) \qquad v = \frac{1}{2i}\left[\phi\left(x + \frac{y}{i}\right) - \phi\left(x - \frac{y}{i}\right)\right] + \frac{1}{2}\left[\psi\left(x + \frac{y}{i}\right) + \psi\left(x - \frac{y}{i}\right)\right].$$
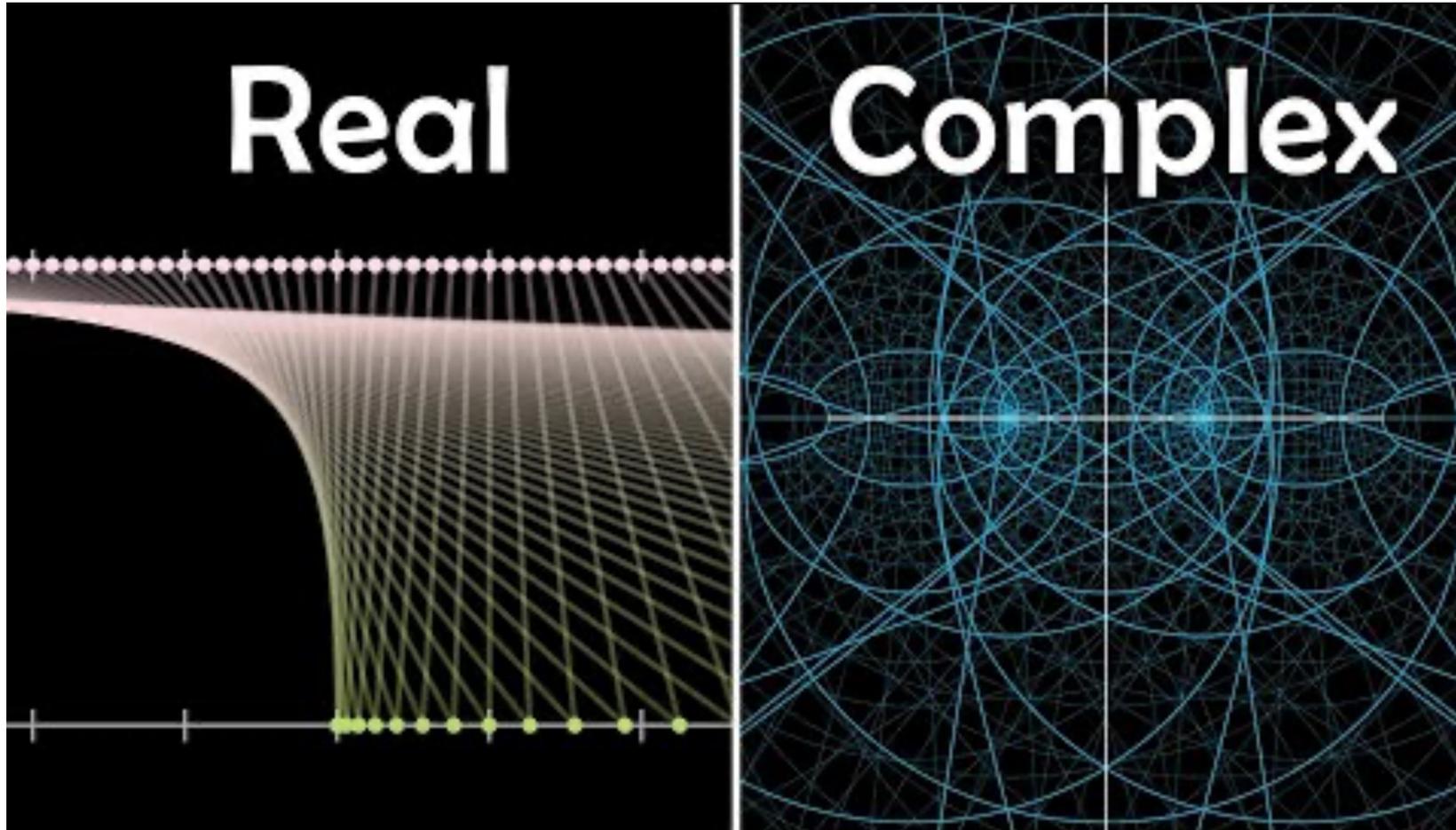
Imaginary parts **"destroy themselves"**

Imaginary parts destroy themselves!

In the course of this work d'Alembert wrote down the equations that later became known as the Cauchy–Riemann equations. However, contrary to what is often stated, it would be **quite inaccurate** from a historical point of view to claim that complex function theory began with d'Alembert's *Essai*. In fact, he **did not connect these equations with any development of complex function theory**. In his hands resorting to "imaginary" expressions was nothing more than a **trick of calculation**. He wanted the integrals of the differential forms that were involved to be real, and at the end of his reasoning he was pleased to remark that the **imaginary quantities had disappeared**. (Bottazzini & Gray, 2010)

Sources and Studies in the History of Mathematics and Physical Sciences

Umberto Bottazzini
Jeremy Gray

Hidden Harmony—
Geometric Fantasies

The Rise of Complex Function Theory

Springer

Complex derivative as an "amplitwist"

https://www.youtube.com/watch?v=0CHZMY02Dhk

Some lessons from Episode 3

- Similarity between the CR equations and the conditions for incompressible/irrotational fluids could have pedagogical potential;

- Complex numbers as "vectors" in 2D only came in the early 1800s, they were not yet a single "entity". The common practice was to manipulate with conjugates to get rid of the imaginary parts.

- When representing vectors in the complex plane, different algebraic rules apply, e.g. multiplication as an amplitwist.

- The Cauchy-Riemann equations should **NOT** be named after d'Alembert.

Complex numbers in Quantum Mechanics

- Complex numbers are useful calculation tools in classical physics, but seem to be *indispensable* in quantum mechanics;

- Why is the wave function complex?

- Griffiths p.1 $i\hbar \dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} + V\Psi.$ p.2 $|\Psi(x,t)|^2\, dx = \left\{ \begin{array}{l} \text{probability of finding the particle} \\ \text{between } x \text{ and } (x+dx), \text{ at time } t. \end{array} \right\}$

- How did Schrödinger perceive this?

What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. ψ is fundamentally a real function (Schrödinger to Lorentz on June 6, 1926)

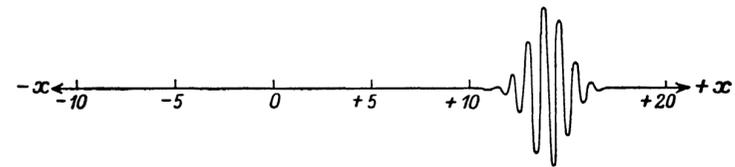## Schrödinger's original struggles with a complex wave function Ⓕ

American Journal of Physics **88**, 433 (2020); https://doi.org/10.1119/10.0000852

Ricardo Karam[a]

Schrödinger's struggles with a complex ψ
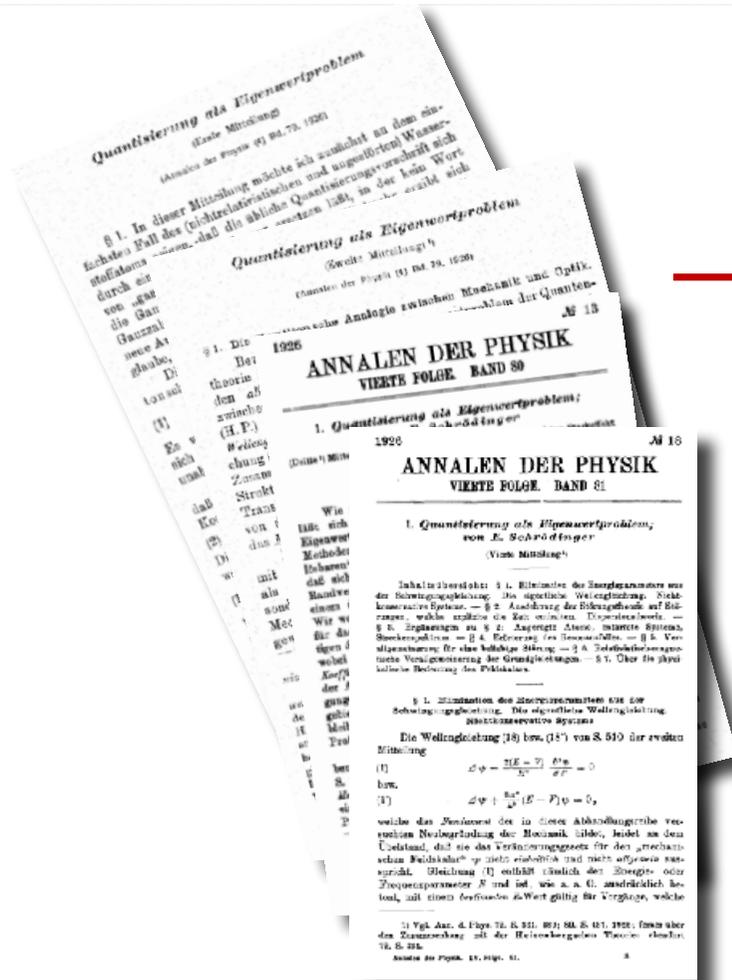
Schrödinger's four communications
(1926)

The Continuous Transition from Micro-
to Macro-Mechanics

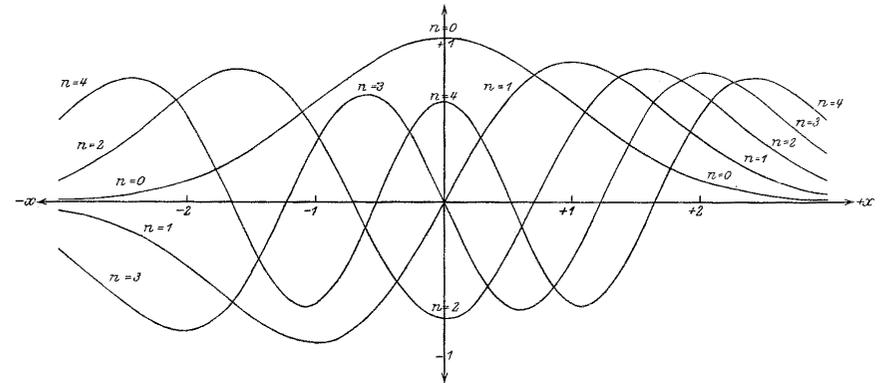(*Die Naturwissenschaften*, 28, pp. 664-666, 1926)

The Continuous Transition from Micro- to Macro-Mechanics

(Die Naturwissenschaften, 28, pp. 664-666, 1926)

Harmonic oscillator

$$\begin{cases} \psi_n = e^{-\frac{x^2}{2}} H_n(x) e^{2\pi i \nu_n t} \\ \left(\nu_n = \frac{2n+1}{2} \nu_0 \; ; \; n = 0, 1, 2, 3... \right) \end{cases}$$

"A group of proper vibrations may represent a particle"

$$\psi = \sum_{n=0}^{\infty} \left(\frac{A}{2} \right)^n \frac{\psi_n}{n!} \qquad \psi = e^{\pi i \nu_0 t - \frac{A^2}{4} e^{4\pi i \nu_0 t} + A x e^{2\pi i \nu_0 t} - \frac{x^2}{2}}$$

Now we take, as is provided for, the real part of the right-hand side

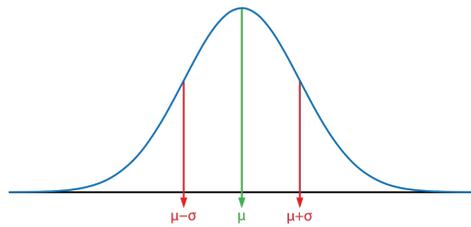[5] i means $\sqrt{-1}$. On the right-hand side the real part is to be taken, as usual.

$$\psi = e^{\frac{A^2}{4} - \frac{1}{2}(x - A \cos 2\pi\nu_0 t)^2} \cos \left[\pi\nu_0 t + (A \sin 2\pi\nu_0 t) . \left(x - \frac{A}{2} \cos 2\pi\nu_0 t \right) \right]$$

$$\psi = \boxed{e^{\frac{A^2}{4}-\frac{1}{2}(x-A\cos 2\pi\nu_0 t)^2}}\boxed{\cos\left[\pi\nu_0 t + (A\sin 2\pi\nu_0 t).\left(x - \frac{A}{2}\cos 2\pi\nu_0 t\right)\right]}$$
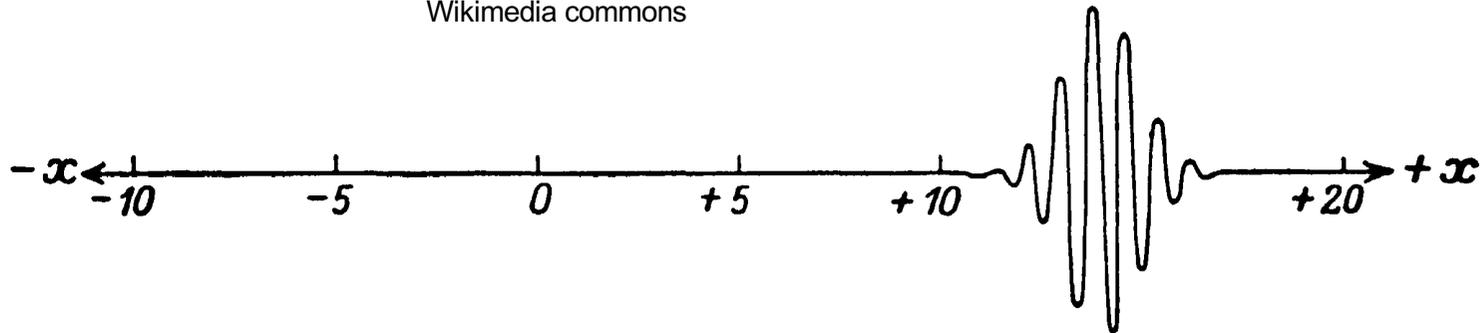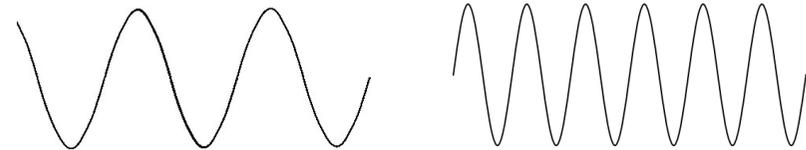
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$
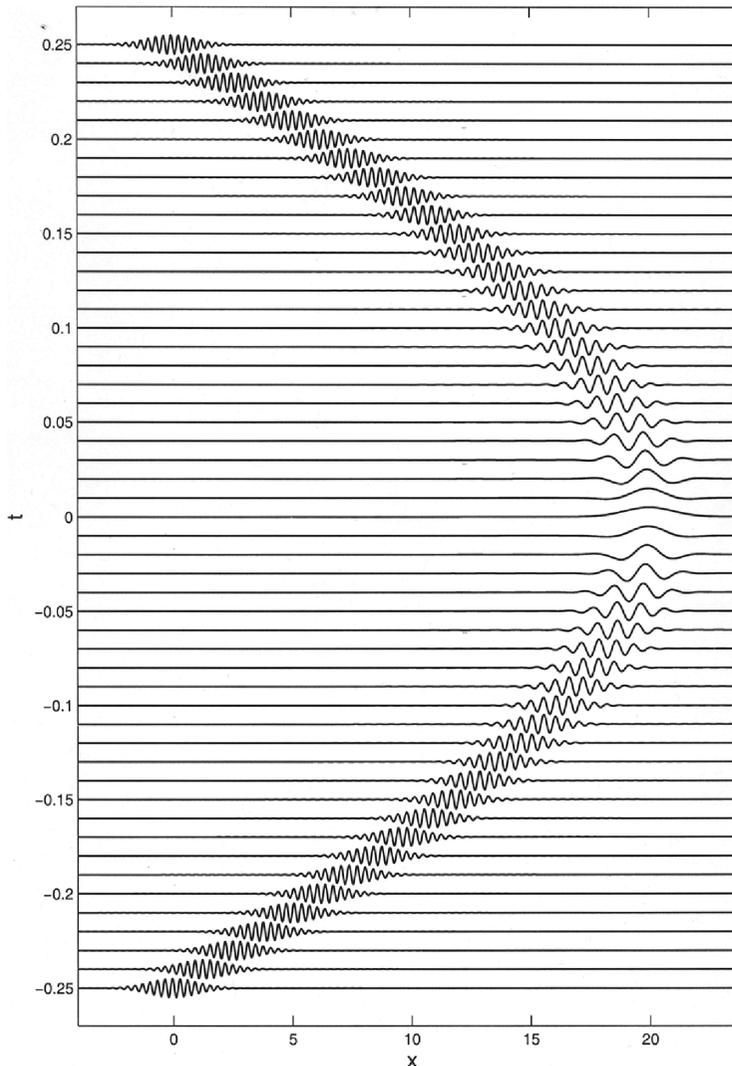
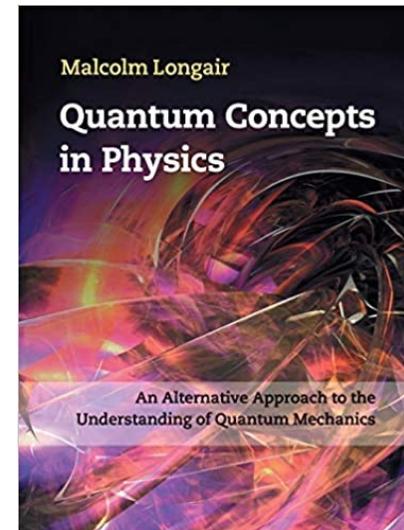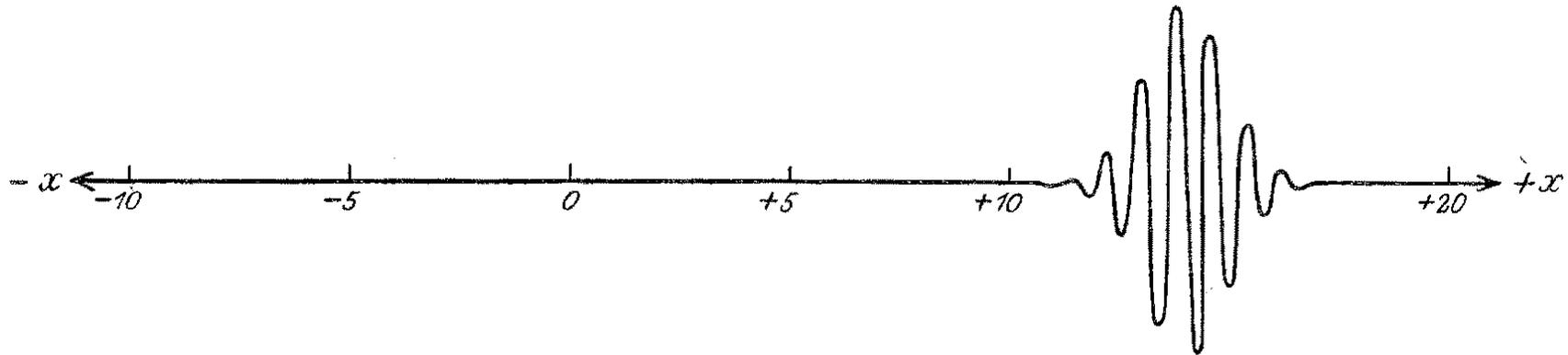Gaussian

cosine wave with varying period

Wikimedia commons

Oscillating wave group as the representation of a particle in wave mechanics

$$\psi = \left[e^{\frac{A^2}{4} - \frac{1}{2}(x - A \quad \sin 2\pi\nu_0 t)} \cdot \left(x - \frac{A}{2} \cos 2\pi\nu_0 t \right) \right]$$

The evolution of wave-packet with A =20 (Diagram created by Dr. David Green)

Source: Chapter 14.6

Malcolm Longair

Quantum Concepts in Physics

An Alternative Approach to the Understanding of Quantum Mechanics

Zoom in: Particles are wave groups

A bold prediction

We can definitely foresee that, in a similar way, wave groups can be constructed which move round highly quantised Kepler ellipses and are the representation by wave mechanics of the hydrogen electron.
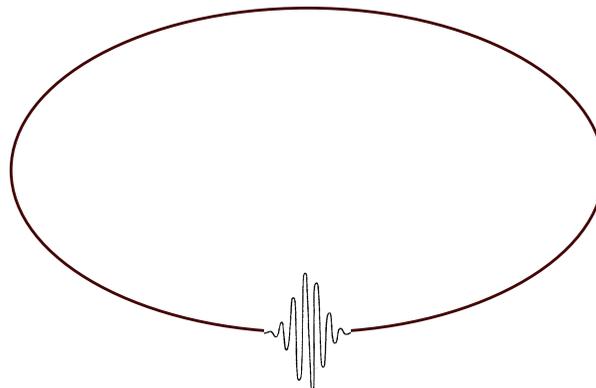
Some lessons from Episode 4

- Schrödinger was initially looking for physical meaning of the *real* component of his wave function.

- An awareness of Schrödinger's original struggles to accept a complex wave function could be comforting for students who are puzzled by a complex *psi*.

- Why not discuss Schrödinger's micro-macro paper when solving the quantum harmonic oscillator in QM courses?

Broader lessons for the Math-Physics interplay

- The math/physics interplay is **complex**; not very useful to talk generally; better look at specific cases and draw situated lessons/conclusions;

- The math/physics interplay is **fruitful**; often one helps the other; but math and physics are **fundamentally different**, and these differences should be made explicit in teaching;

- More could be done to explore the pedagogical potential of the **historical dimension** of the math/physics interplay.

THANKS!

ricardo.karam@ind.ku.dk